End conditions for interpolatory cubic splines with unequally spaced knots

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ABSTRACT

A class of end conditions is derived for cubic spline interpolation at unequally spaced knots. These conditions are in terms of function values at the knots and lead to $0(h^4)$ convergence uniformly on the interval of interpolation.

1. INTRODUCTION

Let s be a cubic spline with knots x_i, $i = 0, 1, \dots, k$, where

$$a = x_0 < x_1 < \dots < x_k = b \tag{1.1}$$

and $h_i = x_i - x_{i-1}$; $i = 1, 2, \dots, k$. Then $s \in C^2[a, b]$ and in each of the intervals $[x_{i-1}, x_i]$, $i = 1, 2, \dots, k$, s is a cubic polynomial.

Given the set of values y_i, $i = 0, 1, \dots, k$, where

$y_i = y(x_i)$; $y \in C^n[a, b]$, $n \geq 4$,

consider the problem of constructing an interpolatory s such that

$$s(x_i) = y_i;\ i = 0, 1, \dots, k. \tag{1.2}$$

Such an interpolatory s can be characterized by the values $m_i = s^{(1)}(x_i)$; $i = 0, 1, \dots, k$, of its first derivative at the knots. If these values are known, s can be constructed in each of the intervals $[x_{i-1}, x_i]$ by use of Hermite's two point interpolation formula. To determine the k + 1 parameters m_i the consistency relations

$$\gamma_i m_{i-1} + 2m_i + \delta_i m_{i+1} = 3\left\{\frac{\gamma_i}{h_i}(y_i - y_{i-1}) + \frac{\delta_i}{h_{i+1}}(y_{i+1} - y_i)\right\}; \quad i = 1, 2, \dots, k-1, \tag{1.3}$$

where

$$\gamma_i = h_{i+1}/(h_i + h_{i+1}) \text{ and } \delta_i = 1 - \gamma_i, \tag{1.4}$$

are used, these being direct consequences of the continuity constraints on s. Since (1.3) provides only k-1 linear equations, it follows that the interpolation conditions (1.2) are not sufficient to determine s uniquely. Two additional linearly independent conditions are always needed for this purpose. These are usually taken to be end conditions, i.e. conditions imposed on s, $s^{(1)}$ or $s^{(2)}$ near the two end points a and b.

As might be expected the choice of end conditions plays a critical role on the quality of the spline approximation. It is well known that the best order of approximation which can be achieved by an interpolatory cubic spline s is

$$\| s - y \| = 0(h^4),$$

where $\| \cdot \|$ denotes the uniform norm on [a, b] and $h = \max_{1 \leq i \leq k} h_i$. Such order of convergence is obtained if, for example, the end conditions

$$m_0 = y_0^{(1)},\ m_k = y_k^{(1)},$$

are used. However, these conditions require knowledge of the first derivative of y at the two end points and, in an interpolation problem, this information is not usually available.

In the present paper we seek to derive end conditions which depend only on the given function values y_i and which give rise to interpolatory cubic splines with $0(h^4)$ convergence uniformly on [a, b]. Such end conditions are those of the $E(\alpha)$ cubic splines considered recently by Behforooz and Papamichael [1]. However, the knots of the $E(\alpha)$ cubic splines are uniformly spaced with $h_i = h$. The purpose of the present paper is to generalise the results of [1] to the case where the knots (1.1) are not equally spaced.

The following two lemmas are needed for the derivation of the results given in section 2.

Lemma 1.1

If $y \in C^4[a, b]$ then, for $x \in [x_{i-1}, x_i]$; $i = 1, 2, \dots, k$,

$$|y^{(r)}(x) - s^{(r)}(x)| \leq a_r h_i^{1-r} \max\{|m_i - y_i^{(1)}|, |m_{i-1} - y_{i-1}^{(1)}|\} + b_r h_i^{4-r} \|y^{(4)}\|;\ r = 0, 1, 2, 3, \tag{1.5}$$

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where

$a_0 = 1/4,\ a_1 = 1,\ a_2 = 6,\ a_3 = 12,$

and

$b_0 = 1/384,\ b_1 = \sqrt{3}/216,\ b_2 = 1/12,\ b_3 = 1/2.$

Lemma 1.2

If $y \in C^5[a, b]$ then, for some $\xi_i \in [x_{i-1}, x_{i+1}]$,

$$\gamma_i \{m_{i-1} - y_{i-1}^{(1)}\} + 2\{m_i - y_i^{(1)}\} + \delta_i \{m_{i+1} - y_{i+1}^{(1)}\} = \beta_i; \qquad i = 1, 2, \dots, k-1, \tag{1.6}$$

where

$$\beta_i = \frac{1}{24} h_i h_{i+1}(h_i - h_{i+1})\, y_i^{(4)} - \frac{1}{60} h_i h_{i+1}(h_i^2 + h_{i+1}^2 - h_i h_{i+1})\, y^{(5)}(\xi_i), \tag{1.7}$$

and γ_i, δ_i are given by (1.4).

Lemma 1.1, which was also used in [1], is a direct consequence of Hermite's two point interpolation polynomial and the optimal error bounds for cubic Hermite interpolation due to Birkhoff and Priver [2]; see also Hall [3]. The result of lemma 1.2 is due to Kershaw [4] and is established by using (1.3) and Peano's method for finding remainders.

2. $E(\alpha_0, \alpha_k)$ CUBIC SPLINES

We let s be a cubic spline interpolating the values $y_i = y(x_i)$; $i = 0, 1, \dots, k$ at the knots (1.1) and, as before we use the abbreviations $h_i = x_i - x_{i-1}$; $i = 1, 2, \dots, k$, $h = \max h_i$ and $m_i = s^{(1)}(x_i)$; $i = 0, 1, \dots, k$. We consider end conditions of the form

$$\left.\begin{aligned} m_0 + \alpha_0 m_1 &= \frac{1}{h_1}\{a_{0\,0}y_0 + a_{1\,0}y_1 + a_{2\,0}y_2 + a_{3\,0}y_3\},\\ \alpha_k m_{k-1} + m_k &= -\frac{1}{h_k}\{a_{0\,k}y_k + a_{1\,k}y_{k-1} + a_{2\,k}y_{k-2} + a_{3\,k}y_{k-3}\}, \end{aligned}\right\} \tag{2.1}$$

and seek to determine the scalars $a_{0\,j}, a_{1\,j}, a_{2\,j}, a_{3\,j}$ and α_j; $j = 0, k$, so that s exists uniquely and

$\| s^{(r)} - y^{(r)} \| = 0(h^{4-r});\ r = 0, 1, 2, 3.$

We do this under the assumption that the knots are arranged so that the ratios

$$h_j/h_1;\ j = 2, 3, \text{ and } h_j/h_k;\ j = k-2, k-1, \tag{2.2}$$

remain constant as $h \to 0$.

Following the technique used in [1], we let

$\lambda_i = m_i - y_i^{(1)};\quad i = 0, 1, \dots, k,$

and assume that $y \in C^5[a, b]$. Then, the equations (1.6) and (2.1) give,

$$\left.\begin{aligned} &\lambda_0 + \alpha_0\lambda_1 = \beta_0,\\ &\gamma_i\lambda_{i-1} + 2\lambda_i + \delta_i\lambda_{i+1} = \beta_i;\ i = 1, 2, \dots, k-1,\\ &\alpha_k\lambda_{k-1} + \lambda_k = \beta_k, \end{aligned}\right\} \tag{2.3}$$

where β_i, $i = 1, 2, \dots, k-1$, are given by (1.7) and

$$\left.\begin{aligned} \beta_0 &= \frac{1}{h_1}\{a_{0\,0}y_0 + a_{1\,0}y_1 + a_{2\,0}y_2 + a_{3\,0}y_3\} - y_0^{(1)} - \alpha_0 y_1^{(1)},\\ \beta_k &= -\frac{1}{h_k}\{a_{0\,k}y_k + a_{1\,k}y_{k-1} + a_{2\,k}y_{k-2} + a_{3\,k}y_{k-3}\} - y_k^{(1)} - \alpha_k y_{k-1}^{(1)}. \end{aligned}\right\} \tag{2.4}$$

As in [1] we assume, without much loss of generality, that $k \geqslant 5$. Then, the equations (2.3) can be written as

$$\lambda_0 + \alpha_0\lambda_1 = \beta_0, \tag{2.5}$$

$$\gamma_2\lambda_1 + 2\lambda_2 + \delta_2\lambda_3 = \beta_2, \tag{2.6}$$

$$\left.\begin{aligned} &c_2\lambda_2 + \tilde{\delta}_2\lambda_3 = \tilde{\beta}_2,\\ &\gamma_i\lambda_{i-1} + 2\lambda_i + \delta_i\lambda_{i+1} = \beta_i;\ i = 3, 4, \dots, k-3,\\ &\tilde{\gamma}_{k-2}\lambda_{k-3} + c_{k-2}\lambda_{k-2} = \tilde{\beta}_{k-2}, \end{aligned}\right\} \tag{2.7}$$

$$\gamma_{k-2}\lambda_{k-3} + 2\lambda_{k-2} + \delta_{k-2}\lambda_{k-1} = \beta_{k-2}, \tag{2.8}$$

$$\alpha_k\lambda_{k-1} + \lambda_k = \beta_k, \tag{2.9}$$

where

$$\left.\begin{aligned} &c_2 = 4 - 2\alpha_0\gamma_1 - \delta_1\gamma_2, && c_{k-2} = 4 - 2\alpha_k\delta_{k-1} - \gamma_{k-1}\delta_{k-2},\\ &\tilde{\delta}_2 = \delta_2(2 - \alpha_0\gamma_1), && \tilde{\gamma}_{k-2} = \gamma_{k-2}(2 - \alpha_k\delta_{k-1}), \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} \tilde{\beta}_2 &= (2 - \alpha_0\gamma_1)\beta_2 - \gamma_2\beta_1 + \gamma_1\gamma_2\beta_0,\\ \tilde{\beta}_{k-2} &= (2 - \alpha_k\delta_{k-1})\beta_{k-2} - \delta_{k-2}\beta_{k-1} + \delta_{k-1}\delta_{k-2}\beta_k. \end{aligned}\right\} \tag{2.11}$$

The matrix of coefficients in (2.3) is the matrix of the linear system which determines the parameters m_i of the spline s. Since (2.3) is equivalent to (2.5)-(2.9), it follows that s exists uniquely provided that the matrix A of the $(k-3) \times (k-3)$ linear system (2.7) is non-singular. This matrix is strictly diagonally dominant if

$$|c_2| > |\tilde{\delta}_2| \text{ and } |c_{k-2}| > |\tilde{\gamma}_{k-2}|. \tag{2.12}$$

It is convenient for the presentation of subsequent results to introduce the following notation,

$$\left.\begin{array}{ll} q_0 = (h_1 + h_2)/h_1, & r_0 = (h_1 + h_2 + h_3)/h_1, \\ q_k = (h_{k-1} + h_k)/h_k, & r_k = (h_{k-2} + h_{k-1} + h_k)/h_k. \end{array}\right] \tag{2.13}$$

Then, using (2.10) and (2.12), it can be easily shown that the matrix A is strictly diagonally dominant for

$\alpha_j < C_j$ or $\alpha_j > D_j$,

where

$$\left.\begin{array}{l} C_j = \{4q_j r_j - 2q_j^2 - q_j - r_j\} / \{(q_j - 1)(2r_j - q_j - 1)\}, \\ D_j = \{4q_j r_j + 2q_j^2 - 5q_j - r_j\} / \{(q_j - 1)(2r_j + q_j - 3)\}\ ; \end{array}\right] \quad j = 0, k. \tag{2.14}$$

To ensure the unique existence of s we assume that the parameters α_j lie in the domains defined by (2.14). Then, using a result of Lucas [5 : 576],

$$\| A^{-1} \|_\infty \leqslant \max \left[\frac{1}{\mu}, 1\right] = \nu \tag{2.15}$$

where $\mu > 0$ is such that

$$|c_2| \geqslant |\tilde{\delta}_2| + \mu \text{ and } |c_{k-2}| \geqslant |\tilde{\gamma}_{k-2}| + \mu .$$

Our assumption concerning the ratios (2.2) implies that the eight coefficients γ_i, δ_i; $i = 1, 2, k-2, k-1$ and the four parameters q_i, r_i, $i = 0, k$ remain constant as $h \to 0$. It follows that the domains (2.14) of α_j and the bound (2.15) of $\| A^{-1} \|_\infty$ do not change with h.

From (1.7)

$$|\beta_i| \leqslant \frac{1}{24} h^2 |h_{i+1} - h_i| \, \| y^{(4)} \| + \frac{1}{60} h^4 \| y^{(5)} \|; \quad i = 1, 2, \ldots, k-1.$$

Thus, from (2.7), (2.11) and (2.15),

$$\left.\begin{array}{l} \max\limits_{2 \leqslant i \leqslant k-2} |\lambda_i| \leqslant \Lambda \\ \text{where} \\ \Lambda = \nu \{\max(|2 - \alpha_0 \gamma_1| + \gamma_2, |2 - \alpha_k \delta_{k-1}| + \delta_{k-2}, 1)\} \\ \quad \{\frac{h^2}{24} \max\limits_i |h_{i+1} - h_i| \, \| y^{(4)} \| + \frac{h^4}{60} \| y^{(5)} \| \} \\ \quad + \nu \max(\gamma_1 \gamma_2 |\beta_0|, \delta_{k-1} \delta_{k-2} |\beta_k|) . \end{array}\right] \tag{2.16}$$

Also, from (2.6) and (2.8),

$$\left.\begin{array}{l} |\lambda_1| \leqslant \frac{1}{\gamma_2} \{(2 + \delta_2)\Lambda + \frac{h^2}{24} |h_3 - h_2| \, \| y^{(4)} \| + \frac{h^4}{60} \| y^{(5)} \| \}, \\ |\lambda_{k-1}| \leqslant \frac{1}{\delta_{k-2}} \{(2 + \gamma_{k-2})\Lambda + \frac{h^2}{24} |h_{k-1} - h_{k-2}| \, \| y^{(4)} \| \\ \quad + \frac{h^4}{60} \| y^{(5)} \| \}, \end{array}\right] \tag{2.17}$$

and, from (2.5), (2.9),

$$\left.\begin{array}{l} |\lambda_0| \leqslant |\alpha_0| \, |\lambda_1| + |\beta_0|, \\ |\lambda_k| \leqslant |\alpha_k| \, |\lambda_{k-1}| + |\beta_k| . \end{array}\right] \tag{2.18}$$

In the above the values of ν, γ_2 and δ_{k-2} remain constant as $h \to 0$.

Thus, lemma 1.1 and the results (2.16)-(2.18) show that

$$\| s^{(r)} - y^{(r)} \| = 0(h^{4-r}), \quad r = 0, 1, 2, 3,$$

only if $\beta_i = 0(h^3)$; $i = 0, k$.

The following theorem generalises the results of theorem 3.1 of [1] to the case of unequally spaced knots.

Theorem 2.1

Let s be an interpolatory cubic spline which agrees with $y \in C^5$ [a, b] at the knots (1.1) and satisfies end conditions of the form (2.1) where the parameters α_j; $j = 0, k$, lie in the domains defined by (2.14). Assume that the knots are arranged so that the ratios (2.2) remain constant as $h \to 0$. Then

$$\| s^{(r)} - y^{(r)} \| = 0(h^{4-r}); \quad r = 0, 1, 2, 3,$$

only if, in (2.1),

$$\left.\begin{array}{l} a_{3j} = \{q_j + (1 - q_j)\alpha_j\} / \{r_j(r_j - 1)(r_j - q_j)\}, \\ a_{2j} = \{r_j + (1 - r_j)\alpha_j\} / \{q_j(q_j - 1)(q_j - r_j)\}, \\ a_{1j} = \alpha_j + 1 - q_j a_{2j} - r_j a_{3j}, \\ a_{0j} = -\{a_{1j} + a_{2j} + a_{3j}\} \quad ; \quad j = 0, k. \end{array}\right] \tag{2.19}$$

Proof

By Taylor series expansion we find that

$\beta_i = 0(h^3)$; $i = 0, k$,

only if the scalars a_{0j}, a_{1j}, a_{2j}, a_{3j}, and α_j; $j = 0, k$, satisfy the relations (2.19). More specifically when the relations (2.19) hold we find that

$$\left.\begin{array}{l} \beta_0 = \frac{h_1^3}{24} \{q_0 r_0 - (q_0 - 1)(r_0 - 1)\alpha_0\} \, y_1^{(4)} + F_0 h^4, \\ \beta_k = -\frac{h_k^3}{24} \{q_k r_k - (q_k - 1)(r_k - 1)\alpha_k\} \, y_{k-1}^{(4)} + F_k h^4, \end{array}\right] \tag{2.20}$$

where

$$|F_j| \leqslant \frac{1}{720} \{|2\alpha_j + 11| + |6\alpha_j - 9| + 32|\alpha_j - 2| + 30\} \, \| y^{(5)} \| ; \quad j = 0, k. \tag{2.21}$$

Definition 2.1

A cubic spline s which interpolates the values y_i; $i = 0, 1, \ldots, k$ at the knots (1.1) and satisfies end conditions of the form (2.1) with a_{0j}, a_{1j}, a_{2j} and a_{3j};

$j = 0, k$, given by (2.19) will be called an $E(\alpha_0, \alpha_k)$ cubic spline.
For any values of α_0 and α_k which lie in the domains (2.14) an $E(\alpha_0, \alpha_k)$ cubic spline s exists uniquely and under the conditions of theorem 2.1,

$\| s^{(r)} - y^{(r)} \| = 0(h^{4-r}); \quad r = 0, 1, 2, 3.$

If the knots (1.1) are equally spaced with $h_i = h$, $i = 1, 2, \ldots, k$, then $q_i = 2$ and $r_i = 3$, $i = 0, k$. If also $\alpha_0 = \alpha_k = \alpha$ then (2.19) gives the values

$$a_{3j} = \frac{1}{6}(-\alpha + 2), \qquad a_{2j} = \frac{1}{6}(6\alpha - 9),$$
$$a_{1j} = \frac{1}{6}(-3\alpha + 18), \qquad a_{0j} = \frac{1}{6}(-2\alpha - 11);$$
$$j = 0, k,$$

which are the coefficients that determine the end conditions of the $E(\alpha)$ cubic splines considered in [1]. Thus, the $E(\alpha)$ cubic splines of [1] occur as the special case

$h_i = h; \qquad i = 1, 2, \ldots, k,$

$\alpha_0 = \alpha_k = \alpha,$

of definition 2.1. We note that in this special case (2.14) gives

$\alpha < 11/3$ or $\alpha > 19/5$,

which is the domain taken in [1] to ensure the unique existence of an $E(\alpha)$ cubic spline.
The corollaries stated below establish various alternative representations for the end conditions of an $E(\alpha_0, \alpha_k)$ cubic spline. They are direct generalizations of results obtained for the case of equally spaced knots in [1], and are established by using standard cubic spline identities and simple properties of divided differences. Their proofs are similar to those used for deriving the corresponding results in [1]. Although some of the algebra involved in the derivation of the present more general results is laborious, the proofs are otherwise elementary and, for this reason, they are omitted.

Corollary 2.1

The end conditions of an $E(\alpha_0, \alpha_k)$ cubic spline can be written as

$$\left.\begin{aligned} & m_0 + \alpha_0 m_1 = p_0^{(1)}(x_0) + \alpha_0 p_0^{(1)}(x_1), \\ & \alpha_k m_{k-1} + m_k = \alpha_k p_{k-3}^{(1)}(x_{k-1}) + p_{k-3}^{(1)}(x_k), \end{aligned}\right\} \quad (2.22)$$

where $p_i(x)$ denotes the cubic polynomial interpolating the values y_i, y_{i+1}, y_{i+2} and y_{i+3} at the points x_i, x_{i+1}, x_{i+2} and x_{i+3}.

Corollary 2.1 shows that the end conditions of an $E(0, 0)$ cubic spline can be written as

$$m_0 = p_0^{(1)}(x_0), \qquad m_k = p_{k-3}^{(1)}(x_k). \qquad (2.23)$$

Since in (2.14) $0 < C_j$, it follows that an $E(0, 0)$ spline exists uniquely for any distribution of the knots. The corollary also shows that an $E(\infty, \infty)$ cubic spline can be interpreted as an interpolatory cubic spline with end conditions

$$m_1 = p_0^{(1)}(x_1), \quad m_{k-1} = p_{k-3}^{(1)}(x_{k-1}). \qquad (2.24)$$

The unique existence of an $E(\infty, \infty)$ cubic spline s can be established easily by considering the linear system for the m_i's derived from the consistency relations (1.3) and the end conditions (2.24). Also, an analysis similar to that used for establishing theorem 3.1 shows that, if $y \in C^5[a, b]$ and the ratios (2.2) remain constant as $h \to 0$,

$\| s^{(r)} - y^{(r)} \| = 0(h^{4-r}); \quad r = 0, 1, 2, 3.$

Corollary 2.2

The end conditions of an $E(\alpha_0, \alpha_k)$ cubic spline can be written as

$$\left.\begin{aligned} & h_2 A_0 s^{(2)}[x_0, x_1, x_2, x_3] + B_0 s^{(2)}[x_0, x_1, x_2] = 0, \\ & h_{k-1} A_k s^{(2)}[x_{k-3}, x_{k-2}, x_{k-1}, x_k] \\ & \quad - B_k s^{(2)}[x_{k-2}, x_{k-1}, x_k] = 0, \end{aligned}\right\} \quad (2.25)$$

where

$$\left.\begin{aligned} & A_j = r_j(r_j - q_j)^2 \{q_j - \alpha_j(q_j - 1)\}, \\ & B_j = (q_j - 1)\{q_j(r_j - 1 + (r_j - q_j)^2) + r_j(q_j - 1) \\ & \quad - \alpha_j(q_j - 1)(r_j - 1 + (r_j - q_j)^2)\}; \quad j = 0, k, \end{aligned}\right\} \quad (2.26)$$

and $f[x_0, x_1, \ldots, x_n]$ denotes the nth divided difference of the function f based at the points x_i; $i = 0, 1, \ldots, n$.
Let

$$\alpha_j^{(1)} = q_j/(q_j - 1); \quad j = 0, k, \qquad (2.27)$$

and

$$\alpha_j^{(2)} = q_j/(q_j - 1) + r_j/\{r_j - 1 + (r_j - q_j)^2\}; \qquad (2.28)$$
$$j = 0, k.$$

Then, corollary 2.2 shows that the end conditions of the $E(\alpha_0^{(i)}, \alpha_k^{(i)})$; $i = 1, 2$, cubic splines can be written respectively as

$$s^{(2)}[x_0, x_1, x_2] = s^{(2)}[x_{k-2}, x_{k-1}, x_k] = 0, \qquad (2.29)$$

and

$$s^{(2)}[x_0, x_1, x_2, x_3] = s^{(2)}[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = 0. \qquad (2.30)$$

Using (2.14) it can be easily shown that

$\alpha_j^{(i)} < C_j; \quad j = 0, k, \quad i = 1, 2.$

Thus, the $E(\alpha_0^{(i)}, \alpha_k^{(i)})$; $i = 1, 2$, cubic splines exist uniquely for any distribution of the knots. When the knots are equally spaced then,

$$\alpha_0^{(1)} = \alpha_k^{(1)} = 2, \quad \alpha_0^{(2)} = \alpha_k^{(2)} = 3,$$

and the end conditions (2.29) and (2.30) become respectively the end conditions

$$\Delta^2 M_0 = \nabla^2 M_k = 0, \tag{2.31}$$

$$\Delta^3 M_0 = \nabla^3 M_k = 0; \quad M_i = s^{(2)}(x_i), \tag{2.32}$$

of the E(2) and E(3) cubic splines considered in [1]. More generally, if the knots are equally spaced and $\alpha_0 = \alpha_k = \alpha$ then, from (2.26),

$$A_j = 3(2-\alpha), \quad B_j = (9-3\alpha),$$

and (2.25) become the end conditions

$$\left.\begin{aligned} &(2-\alpha)\Delta^3 M_0 + (9-3\alpha)\Delta^2 M_0 = 0, \\ &(2-\alpha)\nabla^3 M_k - (9-3\alpha)\nabla^2 M_k = 0, \end{aligned}\right] \tag{2.33}$$

of the $E(\alpha)$ cubic splines considered in [1].
Let d_i denote the jump discontinuity of $s^{(3)}$ at the knot x_i. Then, it can be easily shown that

$$\begin{aligned} d_i &= s^{(3)}(x_i+) - s^{(3)}(x_i-) \\ &= (h_i + h_{i+1})\, s^{(2)}[x_{i-1}, x_i, x_{i+1}]; \end{aligned} \tag{2.34}$$

$$i = 1, 2, \ldots, k-1.$$

The use of (2.34) in conjunction with (2.25)-(2.26) leads to the following corollary.

Corollary 2.3

Let s be an $E(\alpha_0, \alpha_k)$ cubic spline. Then

$$\left.\begin{aligned} &F_0 d_1 = G_0 d_2, \\ &F_k d_{k-1} = G_k d_{k-2}, \end{aligned}\right] \tag{2.35}$$

where

$$\left.\begin{aligned} &F_j = (r_j - 1)\{r_j(1-q_j) + (1-r_j)[q_j - \alpha_j(q_j-1)]\}, \\ &G_j = q_j(r_j - q_j)^2\{q_j - \alpha_j(q_j - 1)\}; \quad j = 0, k, \end{aligned}\right] \tag{2.36}$$

and d_i denotes the jump discontinuity of $s^{(3)}$ at $x = x_i$.
In particular, corollary 2.3 shows that if s is an $E(\alpha_0^{(1)}, \alpha_k^{(1)})$ cubic spline, with the $\alpha_j^{(1)}$ given by (2.27), then $s^{(3)}$ is continuous at x_1 and x_{k-1}. It also shows that if s is an $E(\alpha_0^{(3)}, \alpha_k^{(3)})$ cubic spline, where

$$\alpha_j^{(3)} = q_j/(q_j - 1) + r_j/(r_j - 1); \quad j = 0, k, \tag{2.37}$$

then $s^{(3)}$ is continuous at x_2 and x_{k-2}, and if s is an $E(\alpha_0^{(4)}, \alpha_k^{(4)})$ cubic spline, where

$$\alpha_j^{(4)} = \frac{q_j^2(r_j - q_j)^2 + q_j(r_j - 1)^2 + r_j(r_j - 1)(q_j - 1)}{(q_j - 1)\{(r_j - 1)^2 + q_j(r_j - q_j)^2\}}; \tag{2.38}$$

$$j = 0, k,$$

then the jump discontinuities of $s^{(3)}$ at x_1 and x_{k-1} are respectively equal to the jump discontinuities at x_2 and x_{k-2}. Using (2.14) it can be shown that

$$\alpha_j^{(i)} < C_j; \quad j = 0, k, \quad i = 3, 4,$$

and thus the $E(\alpha_0^{(i)}, \alpha_k^{(i)})$; $i = 3, 4$, cubic splines exist uniquely for any distribution of the knots. When the knots are equally spaced then $\alpha_0^{(3)} = \alpha_k^{(3)} = 3.5$, $\alpha_0^{(4)} = \alpha_k^{(4)} = 3$, and the $E(\alpha_0^{(i)}, \alpha_k^{(i)})$; $i = 3, 4$ splines coincide respectively with the E(3.5) and E(3) cubic splines considered in [1]. More generally, if the knots are equally spaced and $\alpha_0 = \alpha_k = \alpha$ then, from (2.36),

$$F_j = 2(2\alpha - 7), \quad G_j = 2(2-\alpha),$$

and (2.35) becomes the result

$$(7-2\alpha)d_1 = (\alpha - 2)d_2,$$

$$(7-2\alpha)d_{k-1} = (\alpha - 2)d_{k-2},$$

established in [1] for the $E(\alpha)$ cubic splines.

Corollary 2.4

The end conditions of an $E(\alpha_0, \alpha_k)$ cubic spline can be written as

$$\left.\begin{aligned} &(\alpha_0 - 2)M_0 + (2\alpha_0 - 1)M_1 = (\alpha_0 - 2)\, p_0^{(2)}(x_0) \\ &\quad + (2\alpha_0 - 1)\, p_0^{(2)}(x_1), \\ &(2\alpha_k - 1)M_{k-1} + (\alpha_k - 2)M_k = (2\alpha_k - 1)\, p_{k-3}^{(2)}(x_{k-1}) \\ &\quad + (\alpha_k - 2)\, P_{k-3}^{(2)}(x_k), \end{aligned}\right] \tag{2.39}$$

where $M_i = s^{(2)}(x_i)$ and $P_i(x)$ is the cubic polynomial interpolating the values y_i, y_{i+1}, y_{i+2} and y_{i+3} at x_i, x_{i+1}, x_{i+2} and x_{i+3}.
Corollary 2.4 shows that the end conditions of an E(0.5, 0.5) cubic spline can be written as

$$M_0 = p_0^{(2)}(x_0), \quad M_k = p_{k-3}^{(2)}(x_k) \tag{2.40}$$

Since in (2.14) $0.5 < C_j$; $j = 0, k$ it follows that an E(0.5, 0.5) cubic spline exists uniquely for any distribution of the knots.

3. NUMERICAL RESULTS AND DISCUSSION

In this section we present numerical results obtained by taking $y(x) = \exp(x)$,

$$x_0 = 0, \; x_i = \{i - 1/(i+1)\} \times 0.05; \quad i = 1, 2, \ldots, 19, \quad x_{20} = 1, \qquad (3.1)$$

and constructing various $E(\alpha_0, \alpha_{20})$ cubic splines. The splines considered are those which correspond respectively to the values

$$\left.\begin{array}{l} \alpha_0 = \alpha_{20} = 0, \quad \alpha_0 = \alpha_{20} = 0.5, \quad \alpha_0 = \alpha_{20} = \infty, \\ \alpha_j = \alpha_j^{(1)}, \quad \alpha_j = \alpha_j^{(2)} \text{ and } \alpha_j = \alpha_j^{(4)}; \quad j = 0, 20, \end{array}\right\} \quad (3.2)$$

where $\alpha_j^{(1)}$, $\alpha_j^{(2)}$ and $\alpha_j^{(4)}$ are given by (2.27), (2.28) and (2.38) respectively.

The splines corresponding to the first five values of (3.2) are of interest because their end conditions have the particularly simple representations (2.23), (2.40), (2.24), (2.29) and (2.30) respectively. The spline corresponding to the values $\alpha_j^{(4)}$; $j = 0, 20$, is of interest because in the case of equally spaced knots the $E(\alpha_0^{(4)}, \alpha_k^{(4)})$ cubic spline, like the $E(\alpha_0^{(2)}, \alpha_k^{(2)})$ spline, coincides with the $E(3)$ cubic spline of [1]. As was shown in [1] the $E(3)$ cubic spline is the best $E(\alpha)$ spline in the sense that it gives the smallest error bound. (This can also be deduced easily from (2.20) and (2.16) - (2.18)).

The results listed in table 3.1 are values of

$$|\exp(x) - s(x)|$$

computed at various points x between the knots. We also list values computed by constructing the natural cubic spline (N.C.S.), with knots (3.1), interpolating the function $y(x) = \exp(x)$ at the knots. To clarify the presentation we let

$$I_i \equiv (x_{i-1}, x_i); \qquad i = 1, 2, \ldots, 20,$$

and in the table we indicate the interval I_i in which each x lies.

The numerical results indicate the damaging effect that the end conditions

$$s^{(2)}(x_0) = s^{(2)}(x_{20}) = 0, \qquad (3.3)$$

of the natural cubic spline, have upon the quality of the approximation, and demonstrate clearly the considerable improvement in accuracy obtained by using end conditions of the type considered in the present paper instead of (3.3). The results also show that the two splines which correspond respectively to the values $\alpha_j^{(2)}$ and $\alpha_j^{(4)}$; $j = 0, 20$, produce the most accurate approximations.

We recall that the end conditions of the $E(\alpha_0^{(2)}, \alpha_k^{(2)})$ and $E(\alpha_0^{(4)}, \alpha_k^{(4)})$ cubic splines are respectively,

$$s^{(2)}[x_0, x_1, x_2, x_3] = s^{(2)}[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = 0, \qquad (3.4)$$

and

$$\left.\begin{array}{l} (h_1 + h_2)\, s^{(2)}[x_0, x_1, x_2] = (h_2 + h_3)\, s^{(2)}[x_1, x_2, x_3], \\ (h_{k-2} + h_{k-1})\, s^{(2)}[x_{k-3}, x_{k-2}, x_{k-1}] \\ \quad = (h_{k-1} + h_k)\, s^{(2)}[x_{k-2}, x_{k-1}, x_k]. \end{array}\right\} \qquad (3.5)$$

As was remarked previously, when the knots are equally spaced both (3.4) and (3.5) coincide with the end conditions

$$\Delta^3 M_0 = \nabla^3 M_k = 0; \quad M_i = s^{(2)}(x_i),$$

of the $E(3)$ cubic spline which is the best of the $E(\alpha)$ splines considered in [1]. Unfortunately, in the general case of unequally spaced knots the theoretical error bounds (2.16) - (2.18) and (2.20) do not lead readily to the determination of a best $E(\alpha_0, \alpha_k)$ cubic spline. However, the results of table 3.1 as well as results of other numerical experiments support strongly the conjecture that, of all the end conditions considered in the present paper, the conditions (3.4) and (3.5) produce the most accurate approximations.

Table 3.1

$x \in I_i$	x	N.C.S.	E(0, 0)	E(0.5, 0.5)	E(∞, ∞)	$E(\alpha_0^{(1)}, \alpha_{20}^{(1)})$	$E(\alpha_0^{(2)}, \alpha_{20}^{(2)})$	$E(\alpha_0^{(4)}, \alpha_{20}^{(4)})$
I_1	0.0063	0.30×10^{-4}	0.50×10^{-7}	0.46×10^{-6}	0.68×10^{-7}	0.32×10^{-7}	0.43×10^{-8}	0.47×10^{-8}
I_1	0.0188	0.19×10^{-4}	0.34×10^{-7}	0.32×10^{-7}	0.45×10^{-7}	0.23×10^{-7}	0.56×10^{-8}	0.12×10^{-8}
I_4	0.1769	0.85×10^{-6}	0.15×10^{-7}	0.15×10^{-7}	0.15×10^{-7}	0.14×10^{-7}	0.13×10^{-7}	0.13×10^{-7}
I_{10}	0.4702	0.27×10^{-7}	0.27×10^{-7}	0.27×10^{-7}	0.27×10^{-7}	0.27×10^{-7}	0.27×10^{-7}	0.27×10^{-7}
I_{14}	0.6590	0.45×10^{-7}	0.18×10^{-7}	0.18×10^{-7}	0.18×10^{-7}	0.18×10^{-7}	0.18×10^{-7}	0.18×10^{-7}
I_{16}	0.7720	0.16×10^{-5}	0.32×10^{-7}	0.32×10^{-7}	0.31×10^{-7}	0.33×10^{-7}	0.35×10^{-7}	0.36×10^{-7}
I_{19}	0.9224	0.85×10^{-4}	0.22×10^{-6}	0.21×10^{-6}	0.26×10^{-6}	0.17×10^{-6}	0.43×10^{-7}	0.46×10^{-7}
I_{20}	0.9869	0.33×10^{-3}	0.67×10^{-6}	0.65×10^{-6}	0.84×10^{-6}	0.48×10^{-6}	0.83×10^{-8}	0.86×10^{-9}

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